Efficient Integration Algorithms for Cardiac Tissue Simulations: Comparison of Pseudospectral, Spectral, Adaptive Mesh and ADI Methods

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Abstract

We compare the efficiency of several advanced numerical methods for simulating cardiac tissue. The **alternating direction implicit** method is second-order-accurate and allows larger integration time steps than simple explicit methods. **Adaptive mesh refinement** allows coarser spatial resolution and larger times steps in parts of the domain not containing wave fronts while retaining the accuracy of a fine uniform-mesh simulation. **Spectral and pseudospectral** methods allow fewer grid points due to their high-order accuracy. We show representative results and characterize the performance of the methods both in accuracy and execution time for two cell models with different dynamics.

Cell Models

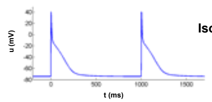
Nygren Model

Model for adult human atrial cells consisting of 12 currents and 29 variables[1].

Ionic currents: I_{Na}, $I_{Ca,L}$, I_t, I_{sus}, $I_{K,r}$, $I_{K,s}$, I_{K1}, I_{CaP}, I_{NaK}, I_{NaCa}, $I_{B,Na}$, $I_{B,Ca}$

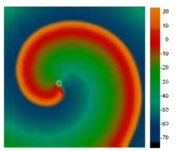
Ion concentrations: $[Na^+]_i$, $[K^+]_i$, $[Ca^{2+}]_i$, $[Na^+]_c$, $[K^+]_c$, $[Ca^{2+}]_c$

Ca^{2+} dynamics: O_C, O_{TC}, O_{TMgC}, O_{TMgMg}, O_{calse}, $[Ca^{2+}]_{rel}$, $[Ca^{2+}]_{up}$, $[Ca^{2+}]_d$



Isolated Cell Action Potential

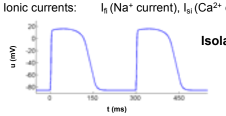
$\partial_t V|_{max}$ = 163.68 (V/s)



3V – Simplified Ionic Model

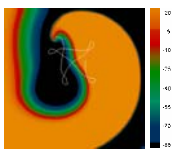
Ionic model consisting of 3 currents and 13 parameters that can be adjusted to reproduce the dynamics and APs of more complicated models. Here it is fitted to reproduce the modified Beeler-Reuter (MBR) model[2].

Ionic currents: I_{fi} (Na^+ current), I_{si} (Ca^{2+} current), I_{so} (K^+ current).



Isolated Cell Action Potential

$\partial_t V|_{max}$ = 162.85 (V/s)



Brief Description of the Methods

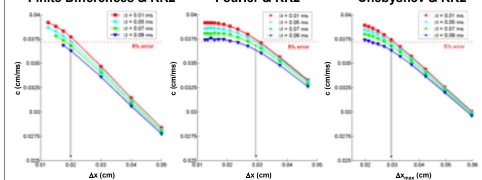
- **Finite Differences**[5]**:** Approximate the derivatives of a function using Taylor series. Only the information of the neighbors is used.
- **Adaptive Mesh**[3]**:** Finite differences, but using finer grids only where the solution is less smooth.

- **Fourier Differentiation**[4,5]**:** The function is approximated by an interpolant in the Fourier basis. The derivatives are taken to be the derivatives of the interpolant.
- **Chebyshev Differentiation**[4]**:** Similar to Fourier, but with Chebyshev polynomials as the basis. The derivatives are taken to be the derivatives of the interpolant.
- **Euler & Runge-Kutta**[5]**:** Explicit time-integration methods. Euler is O(Δt) while the RK used here is O(Δt ²), allowing the use of a larger Δt.
- **ADI**[2,5]**:** In each iteration, one direction of the Laplacian is discretized implicitly, yielding an implicit system of linear equations. This method is also O(Δt ²).

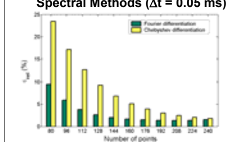
Accuracy Comparison

Conduction velocity is computed as a function of Δx and Δt to determine the maximum Δx and Δt that can be used in each method to achieve a specified accuracy in CV.

Nygren Model

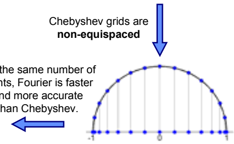


Spectral Methods (Δt = 0.05 ms)

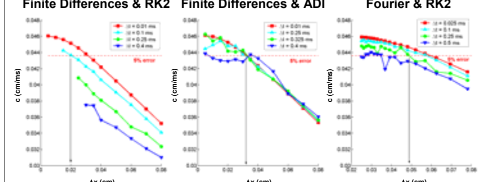


Chebyshev grids are non-equispaced

For the same number of points, Fourier is faster and more accurate than Chebyshev.



3V – MBR Model



Artifactual breakup due to coarse discretization

Δx = 0.025 cm ; Δt = 0.40 ms

This phenomenon occurs for large Δx and/or Δt regardless of method.

Correct solution

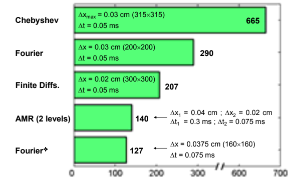
Δx = 0.025 cm ; Δt = 0.35 ms

Time Performance Comparison

Δx and Δt were chosen so that the relative error in conduction velocity $\varepsilon_{cv} \approx 5\%$ for all methods and both cell models.

Nygren Model

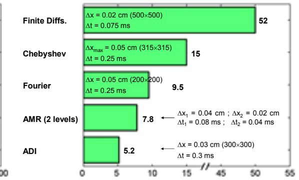
1 second simulation in a 6×6 cm² domain



Execution time (min)

3V – MBR Model

1 second simulation in a 10×10 cm² domain

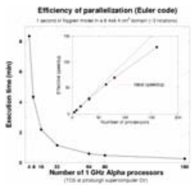


Execution time (min)

✦ Obtaining the same spiral tip trajectory, period and amplitude for the circular core in 2D. However, $\varepsilon_{cv} \approx 10\%$ in maximum conduction velocity.

Conclusions

- **Chebyshev differentiation is not a useful approach for this problem**: the spatial constraint imposed by the sharpness of the wave front forces the solution to be oversolved at the boundaries. Another disadvantage is the difficulty in implementing fiber rotation.
- **Fourier** techniques are found to have the **fastest convergence rate of conduction velocitiy (CV) as a function of Δx**, allowing coarser grids than the other methods.
- **Spectral methods are not as fast as expected** due to the excesively high number of points that are needed to satisfy the spatial constraint.
- For models without stiff detailed calcium dynamics, a larger Δt can be used and an **ADI method** may be especially efficient. On the other hand, solving an implicit system of equations in each iteration with a very small Δt may be computationally expensive.
- Even with only two levels, an **Adaptive Mesh Refinement** is usually one of the most effective methods. This may change if multiple wave fronts are present.
- At present, if the necessary resources are available, **parallelization** gives the best performance (see figure).



References

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